VI Congreso Galego de Estatística e Investigación de Operacións
Vigo 5–7 de Novembro de 2003

STRUCTURAL INERTIA OF VOTING SYSTEMS

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ABSTRACT

Simple games reflect with more or less fidelity the strategic tensions inherent to voting systems. An interesting feature of these systems is their capability to act, *i.e.* their decisiveness. We introduce in this work a normalized measure of the inertia of any simple game from the strictly structural or normative viewpoint. Mathematical properties of this measure are presented, including axiomatic characterizations. The application to a comparative study of certain actual voting systems evidences striking differences as to the inertia degrees they show.

Keywords: simple game, inertia, voting, power.
AMS Classification: 91A12.

1. INTRODUCTION

The design of voting mechanisms applies to many areas of social, political and economical activity. It becomes more and more sophisticated, as long as it pretends to satisfy an increasing number of subtlety requirements. As the real life experience shows, two main tendencies arise. The first one tries to strengthen the *agility* of the mechanism in order to take decisions, and usually applies to national and regional parliaments, town councils, and many other committee systems. The second tendency is rather interested in protecting the rights of certain minorities, even at the cost of introducing a remarkable *inertia* in the mechanism, and it is especially found in supranational organizations. Of course, a basic problem is to conciliate both tendencies. This goal is often difficult to achieve, and it seems therefore interesting to establish objective criteria to evaluate, and hence to compare, the agility/inertia of these decision–making procedures.

The aim of this paper is that of contributing to a better understanding of this question by providing a numerical measure of inertia. This possibility was implicitly suggested by Shapley and Shubik [1954] and only very partially developed by Coleman [1971]. The measure introduced here will be called *decisiveness index*. The decisiveness index applies to every simple game and is closely related to the Banzhaf value defined by Owen [1975]. This relationship proves to be useful for computation purposes.

In the sequel, statements will be given without proof. Proofs can be found in Carreras [2001b], and a copy of this reference can be freely obtained from the author upon request.

2. SIMPLE GAMES AND THE DECISIVENESS INDEX

Definition 2.1. A *(monotonic) simple game* is a pair (N, W), where N is a finite set of *players* and W is a collection of *coalitions* (subsets of N) that satisfies the following properties: (1) $\emptyset \notin W$; (2) if $S \in W$ and $S \subset T$ then $T \in W$ (monotonicity). A coalition S is *winning* if $S \in W$, and *losing* otherwise. Due to monotonicity, the set of *minimal* winnning coalitions W^m determines the game. (For more details on simple games, the interested reader is referred to Carreras [2001a].)

Definition 2.2. Let $\mathcal{SG}$ be the set of all simple games. The *decisiveness index* is the map $\delta : \mathcal{SG} \longrightarrow \mathbb{R}$ given by

$$\delta(N,W) = \frac{|W|}{2^n} \quad \text{for every } (N,W),$$

where $n = |N|$. The number $\delta(N,W)$ will be called the *(decisiveness) degree* of game (N,W). As is obvious, $\delta(N,W) = 0$ iff $W = \emptyset$. Otherwise, $0 < \delta(N,W) < 1$.

Given a game (N,W), the set of coalitions 2^N splits into four classes:

- D (*decisive winning*): class of $S \in W$ such that $N \setminus S \notin W$;
- C (*conflictive winning*): class of $S \in W$ such that $N \setminus S \in W$;
- B (*blocking*): class of $S \notin W$ such that $N \setminus S \notin W$;
- A (*absolutely losing*): class of $S \notin W$ such that $N \setminus S \in W$.

A game (N,W) is *proper* if $C = \emptyset$, and *improper* otherwise; the game is *strong* if $B = \emptyset$, and *weak* otherwise. These two crossed partitions classify all simple games into four types.

Proposition 2.3. Let (N,W) be a nonempty game and (N,W^*) be the dual game. Then:

(a) $\delta(N,W^*) + \delta(N,W) = 1$.

(b) $\delta(N,W^*) - \delta(N,W) = \dfrac{|B| - |C|}{2^n}$. $\square$

Proposition 2.4. Let (N,W) be a nonempty game.

(a) If (N,W) is proper and strong, then $\delta(N,W) = 1/2$.

(b) If (N,W) is proper and weak, then

$$\delta(N,W) = \frac{1}{2} - \frac{|B|}{2^{n+1}} \quad \text{and} \quad \frac{1}{2^n} \leq \delta(N,W) \leq \frac{1}{2} - \frac{1}{2^n}.$$

(c) If (N,W) is improper and strong, then

$$\delta(N,W) = \frac{1}{2} + \frac{|C|}{2^{n+1}} \quad \text{and} \quad \frac{1}{2} + \frac{1}{2^n} \leq \delta(N,W) \leq 1 - \frac{1}{2^n}.$$

(d) If (N,W) is improper and weak, then

$$\delta(N,W) = \frac{1}{2} + \frac{|C| - |B|}{2^{n+1}} \quad \text{and} \quad \frac{1}{2} - \frac{1}{2^n} \leq \delta(N,W) \leq \frac{1}{2} + \frac{1}{2^n}. \quad \square$$

Example 2.5. To illustrate the above results, it will be of interest to look at games with $n \leq 4$ players. Table 1 shows all these games (up to isomorphisms) and their main characteristics. The games are ranked by decreasing degree. The first column gives the ordering number. The second describes the set of minimal winning coalitions (using a simplified notation). The third column classifies each game according to the two crossed partitions: proper (P) or improper (I), and strong (S) or weak (W). All games are weighted majority games, with the sole exception of the three improper and weak games; although no representation is provided, it is not difficult to find in each case. The third column also gives the dual game. The fourth column provides the proportions of the Banzhaf value; to get exact values, it suffices to divide by $2^{n-1} = 8$. Finally, the fifth column gives the decisiveness degree of each game.

Table 1 Simple games with $n \leq 4$ players

Game	W^m	Type/ dual	Banzhaf value	Decisiveness degree
1	{1;2;3;4}	IS / 28	1:1:1:1	0.9375
2	{1;2;3}	IS / 27	2:2:2:0	0.8750
3	{1;2;34}	IS / 26	3:3:1:1	0.8125
4	{1;2}	IS / 25	4:4:0:0	0.7500
5	{1;23;24;34}	IS / 24	4:2:2:2	0.7500
6	{12;13;14;23;24;34}	IS / 23	3:3:3:3	0.6875
7	{1;23;24}	IS / 22	5:3:1:1	0.6875
8	{12;13;14;23;24}	IS / 21	4:4:2:2	0.6250
9	{1;23}	IS / 20	6:2:2:0	0.6250
10	{12;13;14;23}	IS / 19	5:3:3:1	0.5625
11	{1;234}	IS / 17	7:1:1:1	0.5625
12	{12;13;24;34}	IW / 18	3:3:3:3	0.5625
13	{1}	PS / 13	8:0:0:0	0.5000
14	{12;13;23}	PS / 14	4:4:4:0	0.5000
15	{12;13;24}	IW / 15	4:4:2:2	0.5000
16	{12;13;14;234}	PS / 16	6:2:2:2	0.5000
17	{12;34}	IW / 11	3:3:3:3	0.4375
18	{12;13;14}	PW / 12	7:1:1:1	0.4375
19	{12;13;234}	PW / 10	5:3:3:1	0.4375
20	{12;13}	PW / 9	6:2:2:0	0.3750
21	{12;134;234}	PW / 8	4:4:2:2	0.3750
22	{12;134}	PW / 7	5:3:1:1	0.3125
23	{123;124;134;234}	PW / 6	3:3:3:3	0.3125
24	{123;124;134}	PW / 5	4:2:2:2	0.2500
25	{12}	PW / 4	4:4:0:0	0.2500
26	{123;124}	PW / 3	3:3:1:1	0.1875
27	{123}	PW / 2	2:2:2:0	0.1250
28	{1234}	PW / 1	1:1:1:1	0.0625
29	$\emptyset$	PW / 29	0:0:0:0	0.0000

3. MAIN PROPERTIES AND AXIOMATIC CHARACTERIZATION

Our aim in this section is to analyze the behavior of the decisiveness index with regard to the basic forms of compounding simple games and to derive axiomatic characterizations.

Theorem 3.1. Let (N, W) be a game.

(a) If (M, W^M) is the null extension of the game to $M \supset N$, then $\delta(M, W^M) = \delta(N, W)$.

(b) If $S \subset N$ and (N_{-S}, W_{-S}) denotes the residual game when S leaves, then $\delta(N_{-S}, W_{-S}) \leq \delta(N, W)$, and the equality holds iff all players in S are null in (N, W).

(c) If $(N, W) = (N_1, W_1) \times (N_2, W_2) \times \cdots \times (N_r, W_r)$ then

$$\delta(N, W) = \prod_{i=1}^{r} \delta(N_i, W_i).$$

(d) If (N, W') is another game then

$$\delta(N, W \cup W') = \delta(N, W) + \delta(N, W') - \delta(N, W \cap W').$$

(e) If, moreover, the sets E and E' of nonnull players, of (N, W) and (N, W') respectively, are disjoint, then

$$\delta(N, W \cap W') = \delta(N, W)\delta(N, W'). \quad \square$$

In order to obtain an axiomatic characterization of the decisiveness index, we shall consider the following properties:

(A1) *Transfer property*: $\delta(N, W \cup W') = \delta(N, W) + \delta(N, W') - \delta(N, W \cap W')$. This means that the aggregate decisiveness arising from (N, W) and (N, W') is exactly transferred to (*i.e.* shared among) games $(N, W \cup W')$ and $(N, W \cap W')$.

(A2) *Null player property*: If $i \notin N$ and $M = N \cup \{i\}$ then $\delta(N, W) = \delta(M, W^M)$. Neither the adjunction nor the suppression of one or more null players will affect the decisiveness of any game.

(A3) *Unanimity property*: $\delta(N, U_N) = 1/2^n$, where $n = |N|$. This measures the decisiveness of unanimity games.

(A4) *Independence property*: If (N, W) and (N, W') have disjoint nonnull player sets, then $\delta(N, W \cap W') = \delta(N, W)\delta(N, W')$. When combining one game with an 'essentially disjoint' game, the decisiveness of the first game reduces proportionally to that of the second one. This is a particular case of the product property, stated in Theorem 3.1(c).

(A5) *Dictatorship property*: $\delta(\{i\}, U_{\{i\}}) = 1/2$. This is a particular case of the unanimity property.

These five properties can be combined in different ways to provide alternative characterizations of the decisiveness index. Let us see three of them. In each case, the independence of the axiomatic system is clear. And, for not to include a trivial axiom such as $\delta(N, \emptyset) = 0$, we shall restrict ourselves to consider the set $\mathcal{SG}^+$ of all nonempty simple games.

Theorem 3.2. A function $\delta : \mathcal{SG}^+ \longrightarrow \mathbb{R}$ satisfies properties A1, A2 and A3 iff it is the decisiveness index. $\square$

Theorem 3.3. A function $\delta : \mathcal{SG}^+ \longrightarrow \mathbb{R}$ satisfies properties A1, A2, A4 and A5 iff it is the decisiveness index. $\square$

A slight modification of (A5) will give us an axiomatic characterization for the decisiveness index on a fixed player set N. Let us consider

(A5') *Extended dictatorship property*: $\delta(N, U_{\{i\}}) = 1/2$ for all $i \in N$.

Theorem 3.4. Let N be a finite set and $\mathcal{SG}_N^+$ be the set of all nonempty simple games on N. A function $\delta : \mathcal{SG}_N^+ \longrightarrow \mathbb{R}$ satisfies properties A1, A4 and A5' iff it is the restriction of the decisiveness index to $\mathcal{SG}_N^+$. $\square$

4. THE RELATIONSHIP TO THE BANZHAF VALUE

In some manner, and precisely due to its inefficiency, the Banzhaf value β measures the decisiveness of a game from a *local* viewpoint, *i.e.*, from the perspective of each player. The following result, mentioned by Dubey and Shapley [1979] in a slightly different form, establishes the basic relationship between the decisiveness index and the Banzhaf value.

Proposition 4.1. Let (N, W) be a game and $i \in N$. Then

$$\beta_i(N, W) = 2\delta(N, W) - 2\delta(N_{-i}, W_{-i}). \quad \square$$

Proposition 4.1 gives rise to an effective way to compute the decisiveness degree of any game provided that some algorithm to determine the Banzhaf value is available.

Corollary 4.2. Let (N, W) be a game, set $N = \{1, 2, \dots, n\}$, and consider the residual games obtained by successive elimination of players, *i.e.*

$$(N, W)^{-i} = (N_{-\{1,2,\dots,i\}}, W_{-\{1,2,\dots,i\}}) \quad \text{for} \quad i = 1, 2, \dots, n-1.$$

If $\beta^i = \beta_i(N, W)^{-(i-1)}$ for $i = 1, 2, \dots, n$, with $(N, W)^{-0} = (N, W)$, then

$$\delta(N, W) = \frac{1}{2}\Big[\beta^1 + \beta^2 + \cdots + \beta^n\Big]. \quad \square$$

In practice, it will be always convenient to eliminate players according to a ranking of decreasing importance, if possible, in order to ease the procedure at most; this norm is especially indicated for weighted majority games, where the importance can be measured by the weights.

5. TWO EXAMPLES

Two actual instances of decision–making organization will be briefly analyzed here. It is worth mentioning that our approach will be strictly normative, *i.e.* without including ideological or strategic considerations.

Example 5.1. The current European Union (EU) is formed by 15 countries. The most controversial institution within the Union is the Council of Ministers, precisely because it is the battlefield of the agility supporters versus the inertia defenders. The weighted representation is given by:

- Germany, United Kingdom, France and Italy, 10 votes each;
- Spain, 8 votes;
- The Netherlands, Belgium, Portugal and Greece, 5 votes;
- Austria and Sweden, 4 votes;
- Denmark, Finland and Ireland, 3 votes;
- Luxembourg, 2 votes.

A qualified majority of 62 votes out of 87 is required to pass for motions coming from the European Commission (the EU executive body); otherwise, an additional consensus of 10 countries is demanded for approval. When passing from the former system to the latter, the decisiveness degree moves from 0.0778 to 0.0704 and $|W|$ decreases from 2549 to 2307 whereas $|B|$ increases from 27669 to 28154. In percentages, power is quite proportional to weight in both cases.

Example 5.2. As a game among parties, the Spanish Parliament Lower House (the *Congreso de los Diputados*) can be described by the weighted majority game

$$(N, W) \equiv [176; w_1, w_2, \ldots, w_m],$$

where w_j is the number of seats of party j (350 in all). The decisiveness degree and the power distribution heavily depend on the share of seats among parties. Then every legislature must be analyzed separately. Table 2 summarizes six legislative periods.

Table 2 Legislatures in the Spanish Parliament Lower House

Legislature	Party structure (N, W)	Main party	$\delta(N, W)$	$\beta_1(N, W)$
1982–1986	$[176; 202, \ldots]$	PSOE	0.5000	100.00 %
1986–1989	$[176; 184, \ldots]$	PSOE	0.5000	100.00 %
1989–1993	$[176; 175, \ldots]$	PSOE	0.4999	99.76 %
1993–1996	$[176; 159, \ldots]$	PSOE	0.5000	50.00 %
1996–2000	$[176; 156, \ldots]$	PP	0.4966	45.39 %
2000–2004	$[176; 183, \ldots]$	PP	0.5000	100.00 %

6. ACKNOWLEDGEMENTS

Research partially supported by Grant BFM2000–0968 of the Science and Technology Spanish Ministry.

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