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On the extremality of non-standard maximal dual feasible functions

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Abstract

Dual feasible functions were successfully used as a fast tool to get lower bounds for bin packing problems for a long time. Here they are investigated with respect to extremality for the domain $\mathbb{R}$ and also the higher-dimensional domain $[0,1]^m$ with $m \in \mathbb{N}, m > 1$. Both situations are generalizations of the already known domain $[0,1]$.

Keywords: Dual feasible functions; maximality; extremality; general domains

1. Introduction

Dual feasible functions (DFFs) were investigated for a long time, see e. g. [Burdett, Johnson 1977], [Fekete, Schepers 2001], [Carlier, Néron 2007], but only for bounded one-dimensional intervals. These functions can be used to obtain fast lower bounds in cutting stock and packing problems and after an extension to any real arguments also to get valid inequalities for more general linear integer optimization problems.

The m-dimensional vector packing problem (mD-VPP, with $m \in \mathbb{N} \setminus \{0\}$) is a generalization of the well known one-dimensional bin packing problem (1D-BPP). If the dimension of the domain of the DFFs is increased to $[0,1]^m$ then fast lower bounds for the mD-VPP can be obtained, which may be stronger than the bounds arising from the m 1D-BPPs after separating the m directions.

Definition 1. *A function* $f : [0,1] \to [0,1]$ *is a* dual feasible function (DFF)*, if for any finite set* $\{x_i \in \mathbb{R}_+ : i \in I\}$ *of non-negative real numbers, it holds that*

$$\sum_{i \in I} x_i \leq 1 \Longrightarrow \sum_{i \in I} f(x_i) \leq 1. \tag{1}$$

The DFF f *is a* maximal dual feasible function (MDFF)*, if there is no other DFF* g *with* $f(x) \leq g(x)$ *for all* $x \in [0,1]$.

This definition due to [Carlier, Néron 2007] can be adapted to other domains. A MDFF $f : [0,1] \to [0,1]$ is *extreme*, if any DFFs $g, h : [0,1] \to [0,1]$ with $2f(x) = g(x) + h(x)$ for all $x \in [0,1]$ are necessarily identical to f, see [Rietz, Alves, de Carvalho 2010].

If one has to pack items of sizes $\ell_i \in [0,1]$ in the order demands $b_i \in \mathbb{N}$ $(i = 1, \ldots, n)$ into the minimal number of unit bins, then a lower bound for this arises from a DFF $f : [0,1] \to [0,1]$ as $z[f] := \sum_{i=1}^{n} b_i * f(\ell_i)$. Obviously non-maximal DFFs should be avoided to get good bounds, but non-extreme MDFFs are also less useful, because if $g, h : [0,1] \to [0,1]$ are different MDFFs with $2f(x) = g(x) + h(x)$ for all $x \in [0,1]$, then $2z[f] = z[g] + z[h]$, yielding $z[f] \leq \max\{z[g], z[h]\}$, such that either the same result is got or one of the functions g, h dominates f.

2. Extension to domain $\mathbb{R}$, including negative arguments

Definition 2. *A function* $f : \mathbb{R} \to \mathbb{R}$ *is a* dual feasible function (DFF)*, if for any finite set* $\{x_i \in \mathbb{R} : i \in I\}$ *of real numbers, the inequality (1) holds. The DFF* f *is a* maximal dual feasible function (MDFF)*, if there is no other DFF* g *with* $f(x) \leq g(x)$ *for all* $x \in \mathbb{R}$.

The following theorem states necessary and sufficient conditions for a real function to be a MDFF. It summarizes in parts (a)–(c) Theorem 1 and in part (d) Proposition 2 of [Rietz, Alves, de Carvalho, Clautiaux 2012]. The latter part simplifies strongly some proofs, where otherwise longer case distinctions would be needed.

Theorem 1. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a given function.*

(a) If f *satisfies the following conditions then* f *is a MDFF:*

1. $f(0) = 0$*;*
2. f *is superadditive, i.e. for all* $x, y \in \mathbb{R}$*, it holds that*
$$f(x + y) \geq f(x) + f(y); \tag{2}$$
3. *there is an* $\varepsilon > 0$*, such that* $f(x) \geq 0$ *for all* $x \in (0, \varepsilon)$*;*
4. f *obeys the symmetry rule*
$$f(x) + f(1 - x) = 1 \quad \text{for all} \quad x \leq 1/2. \tag{3}$$

(b) If f *is a MDFF then the above properties (1.)–(3.) hold for* f*, but not necessarily (4.).*

(c) If f *satisfies the above conditions (1.)–(3.) then* f *is monotonously increasing.*

(d) If the symmetry rule (3) holds and f *obeys the inequality (2) for all* $x, y \in \mathbb{R}$ *with* $x \leq y \leq \frac{1-x}{2}$ *then* f *is superadditive.*

The following lemma was Proposition 3 in [Rietz, Alves, de Carvalho, Clautiaux 2012]. Its statements about the existence of left and right limits of MDFFs are used in Lemma 4, point 1.

Lemma 1. *If* $f : \mathbb{R} \to \mathbb{R}$ *is a MDFF then for all* $\bar{x} \in \mathbb{R}$ *the limits* $\lim\limits_{x \uparrow \bar{x}} f(x)$ *and* $\lim\limits_{x \downarrow \bar{x}} f(x)$ *exist and* $\lim\limits_{x \uparrow 0} f(x) = \inf\limits_{\bar{x} \in \mathbb{R}} \{\lim\limits_{x \uparrow \bar{x}} f(x) - \lim\limits_{x \downarrow \bar{x}} f(x)\}$*.*

The set of MDFFs with domain $[0, 1]$ is convex, as one can easily prove, but in the case of domain $\mathbb{R}$ the missing symmetry does not allow such a generalization to MDFFs with domain $\mathbb{R}$. Therefore, positive assertions about convex combinations are restricted, like in the following lemma. Its second point will be needed for Proposition 1. Moreover, Lemma 2 allows an adequate definition of extremality of MDFFs also for the domain $\mathbb{R}$.

Lemma 2. *Let* $\lambda \in (0, 1)$ *be a constant and* $g, h : \mathbb{R} \to \mathbb{R}$ *be DFFs. Define the function* $f : \mathbb{R} \to \mathbb{R}$ *as* $f(x) := \lambda g(x) + (1 - \lambda) * h(x)$*. Then the following is true:*

1. f *is a DFF, hence the set of DFFs is convex.*
2. *If* f *is a MDFF then* g *and* h *are that too.*

Proof. 1. For any finite set $\{x_i \in \mathbb{R} : i \in I\}$ of real numbers with $\sum\limits_{i \in I} x_i \leq 1$, one gets $\sum\limits_{i \in I} f(x_i) = \sum\limits_{i \in I} \big(\lambda * g(x_i) + (1 - \lambda) * h(x_i)\big) = \lambda * \sum\limits_{i \in I} g(x_i) + (1 - \lambda) * \sum\limits_{i \in I} h(x_i) \leq \lambda * 1 + (1 - \lambda) * 1 = 1$.
2. Suppose, g is not maximal. Then there are a DFF $\hat{g} : \mathbb{R} \to \mathbb{R}$ with $\hat{g}(x) \geq g(x)$ for all $x \in \mathbb{R}$ and additionally a number $\hat{x} \in \mathbb{R}$ with $\hat{g}(\hat{x}) > g(\hat{x})$. Since $\hat{g}$ and h are DFFs, their convex combination $\hat{f} : \mathbb{R} \to \mathbb{R}$ with $\hat{f}(x) := \lambda \hat{g}(x) + (1 - \lambda) * h(x) \geq f(x)$ for all $x \in \mathbb{R}$ is a DFF. One gets $\hat{f}(\hat{x}) - f(\hat{x}) = \lambda * (\hat{g}(\hat{x}) - g(\hat{x})) > 0$ in contradiction to the prerequisite that f is a MDFF. Similarly h is a MDFF, because $1 - \lambda > 0$. □

In spite of the missing convexity of the set of MDFFs $\mathbb{R} \to \mathbb{R}$, the following definition adapted from [Rietz, Alves, de Carvalho, Clautiaux 2012] is adequate due to Lemma 2:

Definition 3. *A MDFF* $f : \mathbb{R} \to \mathbb{R}$ *is* extreme*, if any DFFs* $g, h : \mathbb{R} \to \mathbb{R}$ *with* $2f(x) = g(x) + h(x)$ *for all* $x \in \mathbb{R}$ *are necessarily identical to* f*.*

Non-extreme MDFFs should be avoided like non-maximal DFFs because of the same reasons as already explained above at the end of Section 1.

Lemma 2 of [Rietz, Alves, de Carvalho 2010] can easily be adapted to domain and range $\mathbb{R}$ with unchanged proof, yielding the following lemma, which may simplify extremality proofs of piecewise linear functions. Its second point will be used for Proposition 1.

Lemma 3. *Let numbers* $a, b \in \mathbb{R}$ *with* $a < b$ *and a MDFF* $f : \mathbb{R} \to \mathbb{R}$ *be given such that for all* $x, y \in [a, b]$*, it holds that* $f(x + y) = f(x) + f(y)$*. Then the following hold:*

1. $f(x) = \frac{f(b)-f(a)}{b-a} * (x - a) + f(a)$ *for all* $x \in [a, b]$*, i.e.* f *is linear in that interval.*
2. *If* $g, h : \mathbb{R} \to \mathbb{R}$ *are MDFFs with* $2f(x) = g(x) + h(x)$ *for all* $x \in \mathbb{R}$ *then the equations* $f(a) = g(a)$ *and* $f(b) = g(b)$ *imply* $f(x) = g(x)$ *for all* $x \in [a, b]$*.*

The first two points of the following lemma were originally points 4 and 5 of Lemma 3 of [Rietz, Alves, de Carvalho 2012] for domain and range $[0, 1]$. In that article, that was used to prove the extremality of functions like $f_{BJ,1}$ (see Proposition 1 below) under certain conditions. Point 3 of Lemma 4 will explicitely be used in Proposition 1, because MDFFs with domain $\mathbb{R}$ are not generally symmetric.

Lemma 4. *If* $f, g, h : \mathbb{R} \to \mathbb{R}$ *are MDFFs with* $2f(x) = g(x) + h(x)$ *for all* $x \in \mathbb{R}$ *then it holds that:*

1. *If* f *is continuous at* $\bar{x} \in \mathbb{R}$ *then* g *and* h *are also continuous at* $\bar{x}$*.*
2. *Let* $a, b \in \mathbb{R}$ *be given numbers. If* $f(a) = f(b)$ *then* $g(a) = g(b)$ *and* $h(a) = h(b)$*.*
3. *If* f *obeys the symmetry rule (3) then* g *and* h *too.*

Proof. Points 1 and 2 can be proved as in [Rietz, Alves, de Carvalho 2012]. The needed limits for the continuity proof exist due to Lemma 1.
1. Let $G_u := \lim_{x \uparrow \bar{x}} g(x)$, $G_o := \lim_{x \downarrow \bar{x}} g(x)$, $H_u := \lim_{x \uparrow \bar{x}} h(x)$ and $H_o := \lim_{x \downarrow \bar{x}} h(x)$. Then $G_u \le g(\bar{x}) \le G_o$ and $H_u \le h(\bar{x}) \le H_o$. Since $2f \equiv g + h$, the continuity of f at $\bar{x}$ implies $\frac{G_u + H_u}{2} = \lim_{x \uparrow \bar{x}} f(x) = f(\bar{x}) = \lim_{x \downarrow \bar{x}} f(x) = \frac{G_o + H_o}{2}$, and hence $G_u = G_o = g(\bar{x})$ and $H_u = H_o = h(\bar{x})$, *i.e.* g and h are continuous at $\bar{x}$.
2. Without loss of generality, let $a \le b$. The monotonicity of the MDFFs g and h implies $g(a) \le g(b)$ and $h(a) \le h(b)$. Since $g(a) + h(a) = 2f(a) = 2f(b) = g(b) + h(b)$, it follows that $g(a) = g(b)$ and $h(a) = h(b)$.
3. One has for any $x \in \mathbb{R}$ that $2 = 2f(x) + 2f(1 - x) = g(x) + h(x) + g(1 - x) + h(1 - x)$ and $g(x) + g(1 - x) \le 1$ and $h(x) + h(1 - x) \le 1$, hence equality holds.
□

The following proposition throws more light on the MDFF $f_{BJ,1}$, due to [Burdett, Johnson 1977], which was already investigated for the domain $[0, 1]$, e.g. in [Clautiaux, Alves, de Carvalho 2010]. This function turned out to yield often good results, if its parameter was chosen optimally, although this function is not extreme under all conditions.

The non-integer part of real expressions is denoted by $\mathsf{frac}(\cdot)$, *i.e.* $\mathsf{frac}(x) \equiv x - \lfloor x \rfloor$ for $x \in \mathbb{R}$.

Proposition 1. *For all* $C \ge 1$*, the following function* $f_{BJ,1} : \mathbb{R} \to \mathbb{R}$ *is an extreme MDFF:*

$$f_{BJ,1}(x) = \Big(\lfloor Cx \rfloor + \frac{\mathsf{frac}(Cx) - \mathsf{frac}(C)}{1 - \mathsf{frac}(C)}\Big)/\lfloor C \rfloor$$

Proof. $f_{BJ,1} : \mathbb{R} \to \mathbb{R}$ is a MDFF obeying the symmetry rule (3) [Rietz, Alves, de Carvalho, Clautiaux 2012]. It remains to show that $f_{BJ,1}$ is extreme. Let $g, h : \mathbb{R} \to \mathbb{R}$ be DFFs with $2f_{BJ,1}(x) = g(x) + h(x)$ for all $x \in \mathbb{R}$. Since $f_{BJ,1}$ is a MDFF, g and h are that too due to Lemma

2, point 2. Therefore, they are superadditive according to Theorem 1, part (b). Since $f_{BJ,1}$ is symmetric, Lemma 4 (point 3) implies the symmetry of the functions g and h.

If $C \in \mathbb{N}$ then $f_{BJ,1}$ is the identity function, hence $2x = g(x)+h(x)$ for all $x \in \mathbb{R}$. Since $g(1) \leq 1$ and $h(1) \leq 1$, one gets $g(1) = h(1) = 1$. Since g and h are MDFFs, one has $g(0) = h(0) = 0$. Lemma 3 (point 2) implies $g(x) = h(x) = x$ for all $x \in [0, 1]$. If $x \in \mathbb{N}$ then taking x times the summand 1 leads to $g(x) \geq x$ and $h(x) \geq x$ due to the superadditivity, hence $g(x) = h(x) = x$. If $x > 1$, but $x \notin \mathbb{N}$ then one gets $x = \lfloor x \rfloor + \mathsf{frac}(x) = g(\lfloor x \rfloor) + g(\mathsf{frac}(x)) \leq g(\lfloor x \rfloor + \mathsf{frac}(x)) = g(x)$ and analogously $h(x) \geq x$, hence $g(x) = h(x) = x$. If $x < 0$ then the symmetry rule (3) completes this part of the proof.

The proof for $C > 2$ and $C \notin \mathbb{N}$ can be done like in Proposition 2 of [Rietz, Alves, de Carvalho 2012], but the variable p in that proof shall now not be bounded, *i.e.* use $p \in \mathbb{Z}$ instead. Moreover, the restriction $x + y \leq 1$ in the discussion of the superadditivity or in the application of Lemma 3 is now not necessary. Therefore, the extremality proof of [Rietz, Alves, de Carvalho 2012] is now also valid for $1 < C < 2$. □

Remark: The proof of [Rietz, Alves, de Carvalho 2012] that $f_{BJ,1}(\cdot; C)$ is not extreme for $1 < C < 2$, if domain and range are restricted to $[0, 1]$, cannot be transferred to domain and range $\mathbb{R}$, because the superadditivity in [Rietz, Alves, de Carvalho 2012] was bound to $x, y \in [0, 1] \wedge x+y \leq 1$. This in the first moment surprising difference is illustrated in the following example.

Example 1. *Let* $C := 9/7$ *and* $x = y = 11/18$. *One gets* $f_{BJ,1}(x; C) = 7/10$, $f_{BJ,1}(x; 2C) = 1/2$, $f_{BJ,1}(2x; C) = 7/5$ *and* $f_{BJ,1}(2x; 2C) = 3/2$, *hence the function* $h := 2f_{BJ,1}(\cdot; C) - f_{BJ,1}(\cdot; 2C)$ *would not be superadditive, since* $h(11/18) = 9/10$ *and* $h(11/9) = 13/10 < 2 * h(11/18)$.

3. Dual feasible functions for vector packing problems

Let $\mathbf{o} := (0, 0, \ldots, 0)^\top \in \mathbb{R}^m$ be the zero vector and $\mathbf{w} := (1, 1, \ldots, 1)^\top \in \mathbb{R}^m$. An instance $E := (n; \mathbf{L}; \mathbf{b})$ of the mD-VPP means that the $n \in \mathbb{N}$ items of sizes $\mathbf{l}_1 := (l_{11}, l_{12}, \ldots, l_{1m}); \ldots; \mathbf{l}_n := (l_{n1}, l_{n2} \ldots, l_{nm})$, given as the rows of the matrix $\mathbf{L}$, have to be put in the order demands $b_1, \ldots, b_n \in \mathbb{N} \setminus \{0\}$ into the minimal number of identical m-dimensional bins, such that the sum of the sizes of the packed items in a bin does not exceed the bin size in any direction. To simplify the presentation, all sizes are assumed to be normalized, *i.e.* the bins are unit cubes and $\mathbf{L} \in [0, 1]^{n \times m}$. The capacity constraint for a pattern $\mathbf{a} \in \mathbb{N}^n$ is then $\mathbf{L}^\top \mathbf{a} \leq \mathbf{w}$ or

$$\sum_{i=1}^{n} a_i * \ell_{id} \leq 1, \; d = 1, \ldots, m. \tag{4}$$

For given vectors $\mathbf{s}, \mathbf{t} \in \mathbb{R}^m$, the relation sign "$\leq$" is used if the relation is componentwise true, *i.e.* $\mathbf{s} \leq \mathbf{t}$ stands for $s_i \leq t_i$, $i = 1, \ldots, m$.

The following definition comes from [Rietz, Alves, de Carvalho, Clautiaux 2013], except that the extremality was no part of that former contribution.

Definition 4. *A function* $f : [0, 1]^m \to [0, 1]$ *is a* vector packing dual-feasible function *(VP-DFF), if for all instances of the mD-VPP and all feasible patterns* $\mathbf{a} \in \mathbb{N}^n$ *satisfying (4), the following inequality holds:*

$$\sum_{i=1}^{n} a_i * f(\mathbf{l}_i^\top) \leq 1.$$

A VP-DFF f *is* maximal *(VP-MDFF), if there is no other VP-DFF* g *with* $g(\mathbf{x}) \geq f(\mathbf{x})$ *for all* $\mathbf{x} \in [0, 1]^m$. *A VP-MDFF* f *is* extreme, *if VP-MDFFs* g *and* h *with* $2f(\mathbf{x}) = g(\mathbf{x}) + h(\mathbf{x})$ *for all* $\mathbf{x} \in [0, 1]^m$ *are necessarily identical with* f.

Since the set of VP-MDFFs is convex [Rietz, Alves, de Carvalho, Clautiaux 2013], this definition of extremality is adequate.

The conditions from the one-dimensional case, stated in [Carlier, Néron 2007], for a function to be maximal are still valid for the higher-dimensional case, as the following theorem shows. This reduced and simplified version compared to [Rietz, Alves, de Carvalho, Clautiaux 2013] will be used for Proposition 3.

Theorem 2. *A function $f : [0,1]^m \to [0,1]$ is a VP-MDFF, if and only if the following holds:*

1. *f is superadditive, i.e. for all $\mathbf{x}, \mathbf{y} \in [0,1]^m$ with $\mathbf{x} + \mathbf{y} \le \mathbf{w}$, it holds that*
$$f(\mathbf{x} + \mathbf{y}) \ge f(\mathbf{x}) + f(\mathbf{y});$$

2. *f is non-decreasing:*
$$f(\mathbf{x}) \le f(\mathbf{y}), \text{ if } \mathbf{o} \le \mathbf{x} \le \mathbf{y} \le \mathbf{w}; \tag{5}$$

3. *f is symmetric, i.e. for all $\mathbf{x} \in [0,1]^m$, it holds that*
$$f(\mathbf{x}) + f(\mathbf{w} - \mathbf{x}) = 1.$$

Some simple VP-DFFs are, for example, the projections to the j^{th} coordinate of the argument-vector ($j = 1, \ldots, m$), *i.e.* $f_j(\mathbf{x}) = x_j$. These functions lead to lower bounds for the mD-VPP which are already known from the 1D-BPP. The following function is a simple example of a non-extreme VP-DFF:
$$f(\mathbf{x}) = \sum_{j=1}^{m} x_j/m = \frac{1}{m}\sum_{j=1}^{m} f_j(\mathbf{x}), \text{ for } m > 1.$$
The following proposition is an example of constructing VP-MDFFs [Rietz, Alves, de Carvalho, Clautiaux 2013]:

Proposition 2. *Let $g : [0,1] \to [0,1]$ be a MDFF and $\mathbf{u} \in \mathbb{R}^m_+$ with $\mathbf{u}^\top \mathbf{w} = 1$. The function $f : [0,1]^m \to [0,1]$ with*
$$f(\mathbf{x}) := g(\mathbf{u}^\top \mathbf{x})$$
is a VP-MDFF.

The resulting functions are extreme under some conditions, but not generally.

Proposition 3. *Let $g : [0,1] \to [0,1]$ be a MDFF, $\mathbf{u} \in \mathbb{R}^m_+$ with $\mathbf{u}^\top \mathbf{w} = 1$, and $f : [0,1]^m \to [0,1]$ the VP-MDFF based on Proposition 2.*

1. *If g is not extreme, then f is not extreme too.*
2. *If g is an extreme staircase function without isolated points $(x; g(x))$ in its graph, then f is extreme.*
3. *If the graph of g contains isolated points $(x; g(x))$, then f may be non-extreme, even if g is extreme.*

Proof. *1.* Since g is not extreme, there are different MDFFs $g_1, g_2 : [0,1] \to [0,1]$ with
$$2g(y) = g_1(y) + g_2(y) \text{ for all } y \in [0,1].$$
Let $f_i(\mathbf{x}) := g_i(\mathbf{u}^\top \mathbf{x})$ for $i = 1, 2$. Clearly, $2f(\mathbf{x}) = f_1(\mathbf{x}) + f_2(\mathbf{x})$, and f_1, f_2 are VP-MDFFs due to Proposition 2. We have to show that $f_1 \not\equiv f_2$. Since $g_1 \not\equiv g_2$, there is a $c \in (0,1)$ with $g_1(c) \ne g_2(c)$. The scalar product $\mathbf{u}^\top \mathbf{x}$ with $x \in [0,1]^m$ reaches all values in $[0,1]$ due to continuity, $\mathbf{u}^\top \mathbf{w} = 1$ and $\mathbf{u}^\top \mathbf{o} = 0$. Hence, there is a vector $\bar{\mathbf{x}} \in [0,1]^m$ with $c = \mathbf{u}^\top \bar{\mathbf{x}}$. Therefore, we have
$$f_1(\bar{\mathbf{x}}) = g_1(\mathbf{u}^\top \bar{\mathbf{x}}) = g_1(c) \ne g_2(c) = f_2(\bar{\mathbf{x}}).$$
2. Assume that there are VP-MDFFs $f_1, f_2 : [0,1]^m \to [0,1]$ with
$$f_1(\mathbf{x}) + f_2(\mathbf{x}) = 2f(\mathbf{x}) = 2g(\mathbf{u}^\top \mathbf{x}) \text{ for all } \mathbf{x} \in [0,1]^m.$$
We have to show that $f_1 \equiv f_2$. First, the argument is restricted to $\lambda \mathbf{w}$, with $\lambda \in [0,1]$. That yields $f(\lambda \mathbf{w}) = g(\lambda \mathbf{u}^\top \mathbf{w}) = g(\lambda)$. The expressions $f_i(\lambda \mathbf{w})$, with $i = 1, 2$, are MDFFs (as functions on λ), because they are obviously superadditive, they have the correct domain and range, and one has
$$f_i((1 - \lambda)\mathbf{w}) = f_i(\mathbf{w} - \lambda \mathbf{w}) = 1 - f_i(\lambda \mathbf{w}),$$

due to the symmetry of f_i. Since g is extreme, the equation $2g(\lambda) = 2f(\lambda\mathbf{w}) = f_1(\lambda\mathbf{w}) + f_2(\lambda\mathbf{w})$ implies $f_1(\lambda\mathbf{w}) = f_2(\lambda\mathbf{w}) = f(\lambda\mathbf{w})$ for all $\lambda \in [0,1]$. It remains to prove that $f_1(\bar{\mathbf{x}}) = f(\bar{\mathbf{x}})$, for any $\bar{\mathbf{x}} \in [0,1]^m$.

Let $\bar{\lambda} := \mathbf{u}^\top\bar{\mathbf{x}}$, and hence $f(\bar{\mathbf{x}}) = g(\bar{\lambda})$. Since g is a staircase function without isolated points in the graph, there are $\lambda_1, \lambda_2 \in [0,1]$ with $\lambda_1 \leq \bar{\lambda} \leq \lambda_2$, $\lambda_1 < \lambda_2$ and $g(\lambda_1) = g(\lambda_2) = g(\bar{\lambda})$. If necessary, the vector $\bar{\mathbf{x}}$ will be replaced by another vector $\mathbf{y} \in [0,1]^m$ with $f(\mathbf{y}) = g(\bar{\lambda})$ in the following way:

A1. Initialize $\mathbf{y} := \bar{\mathbf{x}}$.

A2. While $\mathbf{u}^\top\mathbf{y} > \lambda_1$ and $S_1 := \{r \in \{1,\ldots,m\} : y_r > \bar{\lambda}\} \neq \emptyset$, reduce one or more of these coordinates y_r $(r \in S_1)$ as much as possible, such that $y_r \geq \bar{\lambda}$ remains true for all $r \in S_1$ and additionally $\mathbf{u}^\top\mathbf{y} \geq \lambda_1$ holds.

A3. While $\mathbf{u}^\top\mathbf{y} < \lambda_2$ and $S_2 := \{r \in \{1,\ldots,m\} : y_r < \bar{\lambda}\} \neq \emptyset$, increase one or more of these coordinates y_r $(r \in S_2)$ as much as possible, such that $y_r \leq \bar{\lambda}$ remains true for all $r \in S_2$ and additionally $\mathbf{u}^\top\mathbf{y} \leq \lambda_2$ holds.

A4. If $\mathbf{y} \neq \bar{\lambda}\mathbf{w}$ then go to step A2.

$g(\mathbf{u}^\top\mathbf{y})$ remains unchanged in steps A2 and A3, and hence $f(\mathbf{y})$ too. For any vectors $\mathbf{a}, \mathbf{b} \in [0,1]^m$ with $\mathbf{a} \leq \mathbf{b}$ and $f(\mathbf{a}) = f(\mathbf{b})$, the monotonicity (5) of the VP-MDFFs f_1, f_2 implies $f_i(\mathbf{a}) = f_i(\mathbf{b})$ $(i = 1, 2)$, because $2f(\mathbf{x}) = f_1(\mathbf{x}) + f_2(\mathbf{x})$ for any $\mathbf{x} \in [0,1]^m$. Therefore, $f_i(\mathbf{y})$ did not change in steps A2 and A3. After a finite number of steps the loop terminates, and one gets $f_i(\bar{\mathbf{x}}) = f_i(\bar{\lambda}\mathbf{w}) = f(\bar{\lambda}\mathbf{w})$ for $i = 1, 2$.

3. Let g be the function $f_{FS,1}(x;k)$ due to [Fekete, Schepers 2001] with parameter $k = 2$, *i.e.* $g(x) = \lfloor 2x \rfloor$ for $x \in [0,1] \setminus \{1/2\}$ and $g(1/2) = 1/2$. This function is extreme [Rietz, Alves, de Carvalho 2010]. Let $m = 2$, and $\mathbf{u} := (1/2, 1/2)^\top$. Define the following VP-MDFFs $f_i : [0,1]^2 \to [0,1]$ for $i = 1, 2$:

$$f_i(\mathbf{x}) := \begin{cases} 0, & \text{if } x_1 + x_2 < 1, \\ 1, & \text{if } x_1 + x_2 > 1, \\ x_i, & \text{if } x_1 + x_2 = 1. \end{cases}$$

These functions are superadditive, as it can be easily checked. The symmetry is obvious except for $x_1 + x_2 = 1$. In this case, one gets $f_i(\mathbf{w} - \mathbf{x}) = 1 - x_i = 1 - f_i(\mathbf{x})$, as needed. Therefore, f_1, f_2 are really VP-MDFFs according to Theorem 2. Since f_1, f_2 are different VP-MDFFs with $2f(\mathbf{x}) = f_1(\mathbf{x}) + f_2(\mathbf{x})$ for all $\mathbf{x} \in [0,1]^2$, the function f is not extreme. □

4. Conclusions

The notion of extremality of maximal dual feasible functions can be transferred to MDFFs with more general domains, namely $\mathbb{R}$ and $[0,1]^m$ with $m \in \mathbb{N}, m > 1$, respectively. The function $f_{BJ,1}$ becomes extreme for domain $\mathbb{R}$, also if the parameter was $C \in (1,2)$, while $f_{BJ,1}$ was with domain $[0,1]$ not extreme for the parameter interval $(1,2)$. Under certain conditions, extreme MDFFs can be used also to construct extreme vector packing-maximal dual feasible functions.

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